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Solitons in non-commutative gauge theory

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Abstract: We present a unified treatment of classical solutions of non-commutative gauge theories. We find *all* solutions of the non-commutative Yang-Mills equations of motion in 2 dimensions; and show that they are labeled by two integers — the rank of the gauge group and the magnetic charge. The magnetic vortex solutions are unstable in $2+1$ dimensions, but correspond to the full, stable BPS solutions of $\mathcal{N}=4$ U(1) non-commutative gauge theory in 4 dimensions, that describes N infinite D1 strings that pierce a D3-brane at various points, in the presence of a background B-field in the Seiberg-Witten $\alpha' \to 0$ limit. We discuss the behavior of gauge invariant observables in the background of the solitons. We use these solutions to construct a panoply of BPS and non-BPS solutions of supersymmetric gauge theories that describe various configurations of D-branes. We analyze the instabilites of the non-BPS solitons. We also present an exact analytic solution of non-commutative gauge theory that describes a U(2) monopole.

Keywords: Solitons Monopoles and Instantons, Non-perturbative Effects, D-branes, Brane Dynamics in Gauge Theories.

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1. Introduction

Recently there has been much interest in the properties of non-commutative gauge theories. The interest in these theories was sparked by the discovery that these emerge as limits of M-theory compactifications [1] or of string theory with D-branes in the presence of a background Neveu-Schwarz B-field [2, 3, 4], by the many analogies between non-commutative gauge theories and large-N non-abelian gauge theories [5, 6, 7], and by the many features that non-commutative field theories share with open string theory [7, 8, 9].

In previous papers we constructed and analyzed classical solitons of non-commutative gauge theories [10, 11]. We first constructed exact, BPS, monopole solutions of non-commutative U(1) gauge theory. The solutions were non singular and sourceless, and described smeared monopoles connected to a string-like flux tube [10]. We interpreted this string-monopole as the reflection of a D1 string attached to a D3-brane in the presence of a background Neveu-Schwarz B-field, in the decoupling Seiberg-Witten limit. The calculation of the tension of the string, in precise agreement with that expected from the D1 string, confirmed this picture. In [11] we constructed an extremely simple classical BPS solution of non-commutative U(1) gauge theory that describes an infinite D1 string piercing the D3-brane, which we called the *fluxon*. (See also [12]) We were able to evaluate explicitly the complete spectrum of fluctuations about the fluxon. We found that the fluctuating modes are those of various strings, connected to the D1 string and to the D3-brane.

In this paper we shall present a more unified description of classical solutions of non-commutative gauge theory. We will be studying non-commutative space of two dimensions, namely a space whose coordinates satisfy $[x^i, x^j] = -i\theta^{ij}$, where the antisymmetric matrix θ^{ij} has only two non-vanishing components, say $\theta^{12} = -\theta^{21} = \theta$, although many of our considerations can be easily generalized.

In section 2 we discuss the properties of non-commutative gauge theory. The standard procedure in constructing a non-commutative field theory is to start with an ordinary commutative field theory and replace ordinary products with $\star$ products. In the case of gauge theories one starts with, say, a U(N) gauge theory and replaces ordinary products with $\star$ products in the definition of the field strength, the gauge transformation and the action. Instead we shall proceed more abstractly. The resulting non-commutative gauge theory turns out to include U(N) non-commutative gauge theory *for all values of* N! The value of N will emerge as a superselection parameter. labeling separate sectors of the quantum Hilbert space. We discuss the gauge invariant observables of this theory, the momentum carrying Wilson loops and current densities of fields transforming in the fundamental representation of the gauge group.

In section 3 we shall give a complete classification and construction of all co-dimension two classical solutions of non-commutative gauge theory. These are solutions that are independent of time and all but the two non-commutative spatial directions. They can describe instantons of a euclidean 2-dimensional non-commutative gauge theory, magnetic vortex solitons of a $(2+1)$-dimensional non-commutative gauge theory, or vortex string solitons of a $(3+1)$-dimensional non-commutative gauge theory. Thus they are relevant for discussing D$(p-2)$-branes attached, or immersed in, Dp-branes. The fluxons that we previously considered are special cases of these solutions, but now we identify extra moduli of these solutions that can be identified as the positions of the vortices. We show that the gauge invariant observables of the theory, when calculated in the soliton background, can be used to measure these positions. We also show that, except in the case where the solitons are BPS, the vortices are unstable and can decay by spreading out in the non-commutative space. This is expected for D0- (or D1-) branes immersed in D2- (or D3-) branes.[1] We use these solutions to construct a panoply of BPS and non-BPS solutions of supersymmetric gauge theories that describe various configurations of D-Branes. We analyze the instabilites of the non-BPS solitons.

In section 4 we give an explicit construction of a BPS monopole in non-commutative U(2) Higgsed gauge theory, by solving the non-commutative Nahm equations. This is the non-commutative analogue of the t'Hooft-Polyakov monopole and corresponds precisely, as we show, to a D1 string stretched between two separated D3-branes in the Seiberg-Witten decoupling limit. This is a localized soliton in three dimensions, with an interesting internal spatial structure.

2. Non-commutative gauge theory

2.1 Non-commutative space

Consider $(2+1)$-dimensional space-time with coordinates x^i, $i = 1, 2$ which obey the following commutation relations:

$$[x^i, x^j] = -i\theta^{ij}\,, \qquad \theta^{12} = \theta\,, \qquad [t, x^1] = [t, x^2] = 0\,. \tag{2.1}$$

By non-commutative space-time we mean the algebra $\mathcal{A}_\theta$ generated by the x^i satisfying (2.1). We can think of elements of the algebra as functions of the operators x^i, together with some extra conditions on the allowed expressions in the x^i. We shall largely suppress the dependence on the commutative coordinate t.

It is convenient to introduce the creation and annihilation operators:

$$c^\dagger = \frac{1}{\sqrt{2\theta}}\left(x^1 + ix^2\right), \qquad c = \frac{1}{\sqrt{2\theta}}\left(x^1 - ix^2\right); \qquad [c, c^\dagger] = 1\,. \tag{2.2}$$

The spatial coordinates can then be thought of as operators in the space of Fock states:

$$\mathcal{H} = \{|0\rangle, |1\rangle, \ldots |n\rangle, \ldots\}\,, \tag{2.3}$$

where

$$c|0\rangle = 0\,, \qquad |n\rangle = \frac{c^{\dagger n}}{\sqrt{n!}}|0\rangle\,, \qquad c^\dagger c|n\rangle = n|n\rangle\,. \tag{2.4}$$

Elements of the algebra $\mathcal{A}_\theta$ are then represented as operators in this Fock space, $f(c, c^\dagger)$.

The elements of $\mathcal{A}_\theta$ can also be identified with ordinary functions on $\mathbb{R}^2$, with the product of two functions f and g given by the Moyal formula (or star product):

$$f \star g\,(x) = \exp\left[\frac{i}{2}\theta^{ij}\frac{\partial}{\partial x_1^i}\frac{\partial}{\partial x_2^j}\right] f(x_1)g(x_2)|_{x_1=x_2=x}\,. \tag{2.5}$$

[1] As this work was being completed a discussion of these unstable solitons appeared in [13].

For plane waves:

$$e^{i\vec{p}_1\cdot\vec{x}} \star e^{i\vec{p}_2\cdot\vec{x}} = e^{-\frac{i}{2}\vec{p}_1\times\vec{p}_2} \quad e^{i(\vec{p}_1+\vec{p}_2)\cdot\vec{x}}\,, \tag{2.6}$$

where

$$\vec{p}_1\times\vec{p}_2 = \theta^{ij}p_{1i}p_{2j} = -\vec{p}_2\times\vec{p}_1\,. \tag{2.7}$$

The procedure that maps ordinary commutative functions onto operators in the Fock space is called Weyl ordering and is defined by:

$$\begin{aligned} f(x) = f\left(z = x^1 - ix^2, \bar{z} = x^1 + ix^2\right) &\longmapsto \\ \hat{f}(c, c^\dagger) = \int f(x)\,\frac{d^2x\ d^2p}{(2\pi)^2}\ e^{i\left[\bar{p}_a\left(\sqrt{2\theta}c - z\right) + p_a\left(\sqrt{2\theta}c^\dagger - \bar{z}\right)\right]}&\,, \end{aligned} \tag{2.8}$$

where $p = (p^1 + ip^2)/2$, $\bar{p} = (p^1 - ip^2)/2$. Conversely:

$$f(z,\bar{z}) = \pi\theta\int\frac{d^2p}{(2\pi)^2}\,\mathrm{Tr}\left\{\hat{f}(c,c^\dagger)\ e^{-i\left[\bar{p}_a\left(\sqrt{2\theta}c - z\right) + p_a\left(\sqrt{2\theta}c^\dagger - \bar{z}\right)\right]}\right\}. \tag{2.9}$$

It is easy to see that

$$\text{if}\qquad f\longmapsto\hat{f}\,,\qquad g\longmapsto\hat{g}\qquad\text{then}\qquad f\star g\longmapsto\hat{f}\hat{g}\,. \tag{2.10}$$

We also note that

$$\int d^2x f(x)\longmapsto\pi\theta\mathrm{Tr}\hat{f}(c,c^\dagger)\,. \tag{2.11}$$

The derivative ∂_i is an infinitesimal automorphism of the algebra (2.1):

$$x^i\longrightarrow x^i + \varepsilon^i\,, \tag{2.12}$$

where ε^i is a c-number. For the algebra (2.1) this automorphism is internal:

$$\partial_i f = \frac{d}{d\epsilon^i}f(x^i + \epsilon^i\cdot 1)|_{\epsilon^i=0} = i\theta_{ij}[x^j, f] = [\hat{\partial}_i, f]\,, \tag{2.13}$$

where θ_{ij} is the inverse of θ^{ij}, namely $\theta_{ij}\theta^{jk} = \delta_i^k$, and $\hat{\partial}_i = i\theta_{ij}x^j$. Thus translations in the Fock space are generated by $\hat{\partial}_i = i\theta_{ij}x^j$, so that if $f(x)\mapsto\hat{f}$, then $f(x+a)\mapsto\exp(a\cdot\hat{\partial})\hat{f}\exp(-a\cdot\hat{\partial})$.

2.2 Gauge theory

The standard procedure in constructing a non-commutative field theory is to start with an ordinary commutative field theory and replace ordinary products with $\star$ products. In the case of gauge theories one starts with, say, a U(N) gauge theory and replaces ordinary products with $\star$ products in the definition of the field strength, the gauge transformation and the action. Here we shall proceed more abstractly. The resulting non-commutative gauge theory turns out to include U(N) non-commutative gauge theory *for all values of N.*

Gauge fields arise most naturally via covariant derivatives. In other words, we first consider matter fields, Ψ, which form a representation of the gauge group, or of the gauge algebra, and then form a covariant derivative, ∇_i, so that $\nabla_i\Psi$ is a matter field in the same representation.

The abstract definition of matter fields on a non-commutative space is simply that they are representations of the algebra of non-commutative functions, $\mathcal{A}_\theta$. Thus if f is an element of the algebra then Ψ is a matter field if

$$f : \Psi \longmapsto f \cdot \Psi, \qquad \text{where} \qquad f \cdot (g \cdot \Psi) = (f \star g) \cdot \Psi \,. \tag{2.14}$$

(strictly speaking (2.14) covers only the so-called left modules, there are also right modules, for which: $f \cdot (g \cdot \Psi) = (g \star f) \cdot \Psi$). Given such a representation we then search for a covariant derivative ∇_i that satisfies the Leibnitz rule:

$$\nabla_i(f \cdot \Psi) = (\partial_i f) \cdot \Psi + f \nabla_i \cdot \Psi \,. \tag{2.15}$$

Since the derivative of f satisfies the Leibnitz rule

$$\partial_i(f \star g) = [\hat{\partial}_i, f \star g] = [\hat{\partial}_i, f] \star g + f \star [\hat{\partial}_i, g] = (\partial_i f) \star g + f \star (\partial_i g) \,,$$

this ensures that $\nabla_i\Psi$ is in the same representation of the algebra $\mathcal{A}_\theta$. (Again, this definition of the covariant derivative is specific to algebras like $\mathcal{A}_\theta$ which have enough translational symmetries. We do not need here the more general definition of the gauge field given by Connes [14]).

The simplest representation of the algebra, which is equivalent to operators in the Fock space $\mathcal{H}$, is the Fock space itself, the Fock representation $|\Psi_F\rangle = \sum_{n=0}^{\infty} \Psi_n |n\rangle$. Clearly this is a representation with $f : \Psi_F \mapsto f|\Psi_F\rangle$. What are the possible covariant derivatives? If the Leibnitz rule is satisfied then

$$[\nabla_i, f]|\Psi_F\rangle = i\theta_{ij}[x^j, f]|\Psi_F\rangle = \partial_i f|\Psi_F\rangle \,. \tag{2.16}$$

Consequently

$$\left[\nabla_i - i\theta_{ij}x^j, f\right] |\Psi_F\rangle = 0 \,, \tag{2.17}$$

for any f in the algebra and for any $|\Psi_F\rangle$ in $\mathcal{H}$. The unique solution is that *all gauge fields are of the form:*

$$\nabla_i = i\theta_{ij}x^j + \alpha_i \mathrm{I} = \hat{\partial}_i + \alpha_i \mathrm{I} \,, \tag{2.18}$$

where I is the identity operator and α_i are c-numbers. If we define the connection, or gauge field, to be $A_i = \nabla_i - \hat{\partial}_i$ then this gauge field is given by the trivial

$$A_i = \alpha_i \mathrm{I} \,.$$

The gauge transformations must commute with the action of the algebra in our representation. In our example then these gauge transformation must be given by multiplication by U,

$$|\Psi_F\rangle \longrightarrow U|\Psi_F\rangle \,, \tag{2.19}$$

where U is a c-number. If we demand that the gauge transformations preserve the norm of $|\Psi_F\rangle$, then $U = \exp[i\alpha]$.

The triviality of the gauge transformations and the gauge field follows from the fact that the matter field $|\Psi_F\rangle$ is the analogue of a field with support only at one point in space. To illustrate this point better let us think of the points on an ordinary commutative space X as the irreducible representations of the algebra of functions on this space, A_0. These appear in the decomposition of the algebra viewed as its own representation:

$$f(x) = \int_X \mathrm{d}y\, f(y)\, P_x(y)\,,$$

where $P_x(y) = \delta(x-y)$ is not strictly speaking an element of the algebra of smooth functions, but can be approximated by smooth functions. This relation can be written also as:

$$\mathcal{A}_0 = C^\infty(X) = \bigoplus_{x\in X} R_x\,,$$

where R_x is the one-dimensional irreducible representation of $\mathcal{A}_0$:

$$R_x(f)\cdot\Psi = f(x)\Psi\,.$$

In the same fashion:

$$\mathcal{A}_\theta = \bigoplus_{n\in\mathbb{Z}^+} \mathcal{H}_n\,.$$

where $\mathcal{H}_n$ is a representation of $\mathcal{A}_\theta$ isomorphic to $\mathcal{H}$:

$$f\in\mathcal{A}_\theta\,,\qquad \hat{f}|n\rangle \in \mathcal{H}_n\,.$$

In this sense the Fock representation is the analogue of a single point on the non-commutative space. If we take a direct sum of k copies of $\mathcal{H}$ as another example of representation of $\mathcal{A}_\theta$ then the gauge fields will become $k\times k$ matrices A_i and the gauge transformations will form the unitary group $\mathrm{U}(k)$.

Although translations act non-trivially on $|\Psi_F\rangle$, as $|\Psi_F\rangle \to \exp[a\cdot\hat{\partial}]|\Psi_F\rangle$, this can also be regarded as a gauge transformation, $|\Psi_F\rangle \to f|\Psi_F\rangle$, with $f = \exp[ia^i\theta_{ij}x^j]$.

To construct a matter field defined over all of the non-commutative plane we take

$$|\Psi\rangle = \sum_{nm}\Psi_{nm}|n\rangle\langle m| = \sum_m\left\{\sum_n \Psi_{nm}|n\rangle\right\}\langle m|\,, \tag{2.20}$$

which is a infinite sum of Fock representations, one for each point, $\langle m|$, on the non-commutative plane. In fact Ψ is simply an element of the algebra itself, an operator on the Fock space, and the representation is

$$f:\Psi\longmapsto f\Psi \qquad \text{or} \qquad f:\Psi\longmapsto \Psi f\,.$$

Let us consider the representation $f:\Psi\mapsto\Psi f$, which we shall call the fundamental representation (it is an example of a *right* module; whereas $\Psi^\dagger$ transforms by multiplication on the left and thus forms a left $\mathcal{A}_\theta$-module). Therefore a field in the fundamental representation, is represented by the operator $\sum \psi_{n,m}|n\rangle\langle m|$, where $|n\rangle$ carry all the information about the U(∞) gauge and $\langle m|$ carry all the positional information (vice-versa for $\Psi^\dagger$).

Now we have more freedom in constructing the covariant derivative, in fact

$$\nabla_i \Psi = -i\Psi\theta_{ij}x^j + D_i\Psi\,, \tag{2.21}$$

will satisfy the Leibnitz rule, where D_i is any anti-hermitean operator in the Fock space. Since the ordinary derivative of Ψ is $\partial_i\Psi = [\hat{\partial}_i, \Psi] = [i\theta_{ij}x^j, \Psi]$, we shall write

$$D_i = i\theta_{ij}x^j + A_i\,; \qquad \nabla_i\Psi = [\hat{\partial}_i, \Psi] + A_i\Psi\,. \tag{2.22}$$

Gauge transformations, that preserve the representation, $f : \Psi \mapsto \Psi f$, of the algebra are given by

$$\Psi \longrightarrow \Psi^U = U\Psi\,.$$

To preserve the norm of Ψ we demand that $U^\dagger U = \mathrm{I}$. Consequently, $\Psi^\dagger\Psi$ is a gauge invariant, local, observable. Under such a gauge transformation the covariant derivative of Ψ should transform in the same way as Ψ, so that:

$$D_i\Psi \longrightarrow UD_i\Psi = (UD_iU^\dagger)U\Psi\,. \tag{2.23}$$

Consequently under gauge transformations:

$$D_i \longrightarrow UD_iU^\dagger\,; \qquad A_i \longrightarrow U[\hat{\partial}_i, U^\dagger] + UA_iU^\dagger\,. \tag{2.24}$$

We shall define the field strength, F_{ij}, as usual,

$$F_{ij} = [\nabla_i, \nabla_j] = [D_i, D_j] - i\theta_{ij}\,. \tag{2.25}$$

The covariant derivative D_i transforms in the adjoint representation, like a matter field Φ (an element of the algebra, an operator in the Fock space), whose covariant derivative is

$$\nabla_i\Phi = [D_i, \Phi]\,.$$

Under translations both D_i and Φ transform under a subgroup of the gauge group:

$$(\Phi, D_i) \longrightarrow \exp[ia^i\theta_{ij}x^j](\Phi, D_i)\exp[-ia^i\theta_{ij}x^j]\,,$$

consequently gauge invariant observables constructed from these fields will be translationally invariant.

A gauge invariant bosonic action, or since we supressing the time dependence, an energy density $\mathcal{E}$, can then be formed as

$$\mathcal{E} = \mathrm{Tr}\left\{([D_i, D_j] - i\theta_{ij})^2 + \sum_a \left([D_i, \Phi_a]^2 + V(\Phi_a)\right) + \Psi_a^\dagger\left(D_i^2 + m_a^2\right)\Psi_a\right\}. \tag{2.26}$$

The gauge group under which this density is invariant, generated by (2.19) and (2.24), is that of unitary operators in $\mathcal{H}$, or $\mathrm{U}(\infty)$. Nonetheless, we shall see that, regarded as a functional of D_i, Φ_a and Ψ_a, this expression contains all possible non-commutative gauge theories with gauge group $\mathrm{U}(N)$, for all $N = 0, 1, 2, \ldots$. To see this let us ignore the matter fields and write the action for the $(2+1)$-dimensional non-commutative gauge theory, in the gauge $A_t = 0$, as:

$$S = \frac{2\pi\theta}{4g^2} \int \mathrm{d}t \mathrm{Tr} \left\{ \partial_t D \partial_t \bar{D} - 4 \left[[\bar{D}, D] + \frac{1}{2\theta} \right]^2 \right\}, \tag{2.27}$$

where we have rewritten the covariant derivatives as:

$$D = -\frac{c^\dagger}{\sqrt{2\theta}} + \frac{1}{2}(A_1 + iA_2)\,, \qquad \bar{D} = \frac{c}{\sqrt{2\theta}} + \frac{1}{2}(A_1 - iA_2) = -D^\dagger\,, \tag{2.28}$$

so that the field strength is given by:

$$F = F_{1,2} = 2\left[[\bar{D}, D] + \frac{1}{2\theta} \right]. \tag{2.29}$$

The physics of this system is given by the infinite-dimensional space $\mathcal{F}$ of the operators $D, \bar{D}$ acting in the Fock space $\mathcal{H}$ moded out by the action of the gauge group $\mathcal{U}$ of unitary operators in $\mathcal{H}$, acting via:

$$D \longmapsto UDU^\dagger\,, \qquad \bar{D} \longmapsto U\bar{D}U^\dagger\,, \qquad UU^\dagger = U^\dagger U = 1\,. \tag{2.30}$$

We now argue that this quantum mechanical system describes $\mathrm{U}(N)$ non-commutative gauge theory for all values of N, where N is a superselection parameter. The argument is that for the energy of a field configuration to be finite, or for the action to be finite, F must vanish almost everywhere — i.e. except for a finite number of matrix elements we must have that

$$[D^\dagger, D] = \frac{1}{2\theta}\,.$$

This is obviously true of the absolute minima of the action —- the classical vacua. The unique irreducible representation of this Heisenberg algebra is, up to unitary equivalence,

$$D = -\frac{c^\dagger}{\sqrt{2\theta}}, \qquad D^\dagger = -\frac{c}{\sqrt{2\theta}}\,,$$

and the most general representation, classical vacuum, is a reducible sum of N such irreducible representations, acting in the direct product of N copies of $\mathcal{H}$, $\mathcal{H} = \mathcal{H} \oplus \mathcal{H} \oplus \cdots \mathcal{H} \approx \mathcal{H} \otimes \mathbb{C}^N$, which is isomorphic to $\mathcal{H}$ itself (by a version of the "Hilbert hotel" argument). We label the basis vectors of this space by $|n, a\rangle, a = 1, \ldots, N$, and in this basis:

$$D^{(N)} = -\frac{c^\dagger}{\sqrt{2\theta}} \otimes \mathrm{I}_N\,,$$

$$\text{where} \qquad c^\dagger |n, a\rangle = \sqrt{n+1}|n+1, a\rangle\,, \qquad \mathrm{I}_N |n, a\rangle = |n, a\rangle\,. \tag{2.31}$$

This vacuum is invariant under a $\mathrm{U}(N)$ gauge transformations that act on the a labels. N is an index. It is equal to the difference of the number of zero eigenvalues of the hermitean operators $D^\dagger D$ and $DD^\dagger$, whose non-zero eigenvalues coincide.

As far as we can ascertain the quantum theory constructed about one of these vacua will not mix with the others. Any path in field space that connects different vacua has infinite energy and action. Thus the functional integral for the partition function with action given by (2.27) breaks up into a sum of partition functions for each $\mathrm{U}(N)$ gauge theory:

$$Z = \int \frac{\mathcal{D}D\mathcal{D}\bar{D}}{\mathrm{Vol}\,\mathrm{U}(\infty)} \;\exp[iS] = \sum_{N=0}^{\infty} Z_N\,. \tag{2.32}$$

In the sector labeled by N we would expand $D = D^{(N)} + A^{(N)}$, and would find that the field strength takes the customary form for the field strength of a $\mathrm{U}(N)$ non-commutative gauge theory:

$$F = \frac{1}{\sqrt{2\theta}}\left([c\cdot I, \mathbf{A}] + [c^\dagger \cdot I, \bar{\mathbf{A}}]\right) + [\bar{\mathbf{A}}, \mathbf{A}]\,,$$

where $\mathbf{A}_{ab}$ is an $N \times N$ matrix operator in ordinary Fock space, given in terms of $A^{(N)}$ as:

$$\langle m|\mathbf{A}_{ab}|n\rangle = \langle m, a|A^{(N)}|n, b\rangle\,.$$

It is fascinating that the action for non-commutative gauge theory does not determine the rank of the gauge group, but rather that it emerges as a superselection parameter.

2.3 Gauge invariant observables

What are the physical observables of this theory? They, of course, should be invariant under the $\mathrm{U}(\infty)$ gauge transformations (2.30). Since translation of the non-commutative coordinates are generated, up to shifts of the gauge field, by these unitary transformations, there appears to a conflict between gauge invariance and locality. The simplest gauge invariant observables are in fact non local.

For any $l \in \mathbb{C}$, $l = l_1 + il_2$, consider the operator

$$\mathcal{D}(l) = \bar{l}D + lD^\dagger = l_1 D_1 + l_2 D_2\,.$$

This is a hermitean operator whose eigenvalues are gauge invariant functions on $\mathcal{F}$. Consequently, traces of ordered products of exponentials of $i\mathcal{D}(l_a)$ for different l_a:

$$\mathcal{W}(\vec{l}) = \mathrm{Tr}\,\prod\nolimits_a \exp i\mathcal{D}(l_a)\,, \tag{2.33}$$

provide a set of gauge-invariant functions on $\mathcal{F}$. The functionals (2.33) are the non-commutative analogues of the familiar Wilson loops [15, 16], that describe parallel transport along the path described by the series of displacements along l_a. Unlike the commutative case the "path" along which the loop is taken does not need to be closed: $\sum_a l_a \neq 0$ in general, and $\mathcal{W}(\vec{l})$ is not a local operator — rather it has momentum equal to $\theta^{-1}_{ij}\sum_a l^j_a$.

Another set of gauge-invariant observables can be constructed in terms of the operators D and $\bar{D}$, or in terms of bilinears in fields that transform in the fundamental representation. Consider the space of normalizable solutions of the massless Dirac equation in the background gauge field A_1, A_2. We take the fermions to transform in the fundamental representation of the gauge group,

$$\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} \longmapsto U \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}. \tag{2.34}$$

Then the Dirac equations are, in the operator formalism:

$$D\psi^i_L + \psi^i_L c^\dagger = 0\,, \qquad \bar{D}\psi^j_R - \psi^j_R c = 0\,, \tag{2.35}$$

where $i = 1, \ldots, n_L$ and $j = 1, \ldots, n_R$ and n_L, n_R are the numbers of the left-moving and the right-moving zero modes. These equations are clearly invariant under (2.34). Under translations they transform as

$$\psi_{R,L} \longrightarrow e^{a\cdot\hat{\partial}}\psi_{R,L}e^{-a\cdot\hat{\partial}}\,. \tag{2.36}$$

As we noted before, translations are equivalent, up to gauge transformations, to transformations of the type $\psi_{R,L} \to \psi_{R,L}e^{-a\cdot\hat{\partial}}$.

The components of the $U(n_L)_L \times U(n_R)_R$ current:

$$j_L = \psi^\dagger_L\psi_L\,, \qquad j_R = \psi^\dagger_R\psi_R\,, \tag{2.37}$$

as well as of the "density"

$$\rho = \psi^\dagger_L\psi_R\,, \tag{2.38}$$

are gauge invariant observables of the non-commutative gauge theory. Unlike the Wilson loops these are *local* observables. Under translations the transform as $O \to e^{a\cdot\hat{\partial}}Oe^{-a\cdot\hat{\partial}}$, where O stands for j_R, j_L or ρ.

3. Classical static solutions

In [11] we constructed classical solutions of non-commutative gauge theories, fluxons. In the supersymmetric $3+1$ non-commutative Yang-Mills theory gauge theory these described D1 strings that pierced a D3-brane and were BPS solutions. We also pointed out that vortex line solutions of $3+1$ non-commutative Yang-Mills theory gauge theory or point vortex solutions of $2+1$ non-commutative Yang-Mills theory could be easily generated by setting the scalar field, Φ, that described the extension of the D1 string into the bulk, to zero. We shall now construct all static solutions of pure $2+1$ non-commutative Yang-Mills theory with finite energy. We shall recover the analogue of the N-fluxon solution, however with additional moduli, that we will see describe the position of the vortices. From now on we shall often set $2\theta = 1$, to simplify the formulae. θ can always be reintroduced by scaling the non-commutative coordinates as $x^i \to \sqrt{2\theta} x^i$.

For time independent gauge fields the equations of motion are

$$[D, [\bar{D}, D]] = [\bar{D}, [\bar{D}, D]] = 0\,, \tag{3.1}$$

and the energy is proportional to $\mathrm{Tr}\left([D^\dagger, D] - 1\right)^2$.

3.1 Classification of solutions

Consequently we need to find a pair of operators $D, D^\dagger = -\bar{D}$, acting in $\mathcal{H}$, that obey:

$$\begin{aligned} [D^\dagger, D] &= 1 + F\,, \\ [D, F] &= [D^\dagger, F] = 0\,, \\ \mathrm{Tr}\, F^2 &< \infty\,. \end{aligned} \tag{3.2}$$

The first three equations imply that $D, D^\dagger, 1+F$ form a Heisenberg algebra, with F generating the center of the algebra. The Hilbert space $\mathcal{H}$ decomposes into irreducible representations of this algebra, with F equal to a constant f_n on the n-th component. Let d_n be the dimension of the n-th irreducible component. It is well known that unless $1 + f_n = 0$ it must be that $d_n = \infty$. The finite energy condition implies that:

$$\sum_n d_n f_n^2 < \infty\,. \tag{3.3}$$

Therefore there are just two possibilities:

$$f_n = 0\,, \qquad d_n = \infty \qquad \text{or} \qquad f_n = -1\,, \qquad d_n \geq 0\,. \tag{3.4}$$

Hence, by a unitary gauge transformation, we can bring D and $D^\dagger$ to the following form: on a finite-dimensional subspace V_q of dimensionality q:

$$[D, D^\dagger] = 0\,, \qquad D = \mathrm{diag}\,(-\lambda_1, \ldots, -\lambda_q)\,,$$

(We have chosen this sign convention so that, as we will see below, λ_i will be the position of the i-th vortex, for $2\theta = 2$) while on the complement, $\mathcal{H} \ominus V_q$, which is isomorphic to $\mathcal{H}$, D is a reducible sum of N irreducible representations of the Heisenberg algebra ($D = -c^\dagger, \bar{D} = c$), as given explicitly in (2.31).

Let us, for simplicity, choose $N = 1$, and let ${S_q}^\dagger : \mathcal{H} \ominus V_q \to \mathcal{H}$ be the unitary isomorphism between the two Hilbert spaces. We can extend ${S_q}^\dagger$ to the whole of $\mathcal{H}$ by having it act as 0 on V_q. Then ${S_q}^\dagger$, as an operator in $\mathcal{H}$, obeys:

$${S_q}^\dagger S_q = 1\,, \qquad S_q {S_q}^\dagger = 1 - P_q\,, \tag{3.5}$$

where $P_q : \mathcal{H} \to V_q$ is the orthogonal projection. Again, by unitary gauge transformation we can assume that V_q is spanned by the vectors $|0\rangle, \ldots, |q-1\rangle$. Thus, for $N = 1$, the generic static solution of the non-commutative Yang-Mills equations of motion is given by:

$$D = \lambda_q - S_q c^\dagger S_q{}^\dagger \,, \tag{3.6}$$

where $\lambda_{\mathbf{q}} = \sum_{i=0}^{q-1} \lambda_i |i\rangle\langle i|$, and $S_q|n\rangle = |n+q\rangle$. This gauge field has field strength:

$$F = -P_q \,, \tag{3.7}$$

which implies that the solution has a magnetic charge q. The moduli $\lambda_{\mathbf{i}}$ describe the positions of the vortices, as we shall explicitly see below, by examining the behavior of the Wilson loop or the position dependent fermion bilinears in this background.

Thus all classical static solutions of $(2+1)$-dimensional non-commutative gauge theory are classified by the rank of the gauge group, N, and by the magnetic charge, q. It is at first a little bit surprising to discover solutions of positive charge, and to have no solutions of negative charge. However, the noncommutativity breaks the left-right symmetry and as a consequence, one cannot simply by change of orientation produce anti-vortices from vortices.

It is also surprising to find that there is a $2q$-dimensional moduli space of solutions for magnetic charge q corresponding to the separations of the vortices. Since the vortices are not BPS solutions we would have expected them to repel. Indeed in a commutative gauge theory we could have considered two vortices very far way from each other, and then the one gauge boson exchange interaction could have been calculated exactly —- leading to repulsion. However, in the non-commutative gauge theory the energy density is not gauge invariant and indeed the energy density of the vortices, proportional to $\mathrm{Tr}\left([D, D^\dagger] + 1\right)^2$, is independent of $\lambda_{\mathbf{i}}$, and no similar conclusion can be drawn.

3.2 Fermion condensates

Let us now consider the behavior of fermions, transforming in the fundamental representation, in the presence of the multi-vortex solutions. As we will see below, fermion bilinear observables, such as (2.37) and (2.38), will be sensitive to the moduli λ_i, i.e. to the position of the vortices.

The operator D acting in $\mathcal{H}$, being close to $-c^\dagger$ has a finite spectrum of normalizable eigenstates. Let $-\epsilon_i$, $i = 1, \ldots, n_L$ be the corresponding eigenvalues, and $\chi_i \in \mathcal{H}$ the corresponding eigenstates. Then $\psi_L^i = \chi_i \otimes \langle \epsilon_i|$ is a Dirac zeromode, with $\langle \epsilon_i|$ being the coherent state

$$\langle \epsilon_i| \equiv \langle 0|e^{\epsilon_i c} e^{-\frac{1}{2}\bar{\epsilon}\epsilon} \,.$$

Then the corresponding current j_L is diagonal and has the components:

$$j_L^{\bar{i}i} = |\bar{\epsilon}_i\rangle\langle \epsilon_i| \,.$$

On the other hand if we examine the right handed modes ψ_R^i, we will not find zero modes at all, since the analogue of $\langle\epsilon_i|$ would be a state $\langle\sigma|$ that would be an eigenstate of c, i.e. $\langle\sigma|c = \epsilon\langle\sigma|$, and there are no such normalizable states.

Consider the Dirac equation in the background of the q-vortex solution (3.6). We first look for eigenvectors of D, $\chi_i \in \mathcal{H}$, satisfying

$$D\,\chi_i \;=\; \left[-\lambda_q - S_q c^\dagger {S_q}^\dagger\right]\,\chi_i \;=\; \epsilon_i\chi_i\,.$$

It is clear that there exist q (and no more) solutions of this equation, where

$$\epsilon_i = -\lambda_i\,, \qquad \chi_i = |i\rangle\,, \qquad i = 0,\ldots,q-1\,. \tag{3.8}$$

So $n_L = q$, and

$$\Psi_L^i = |i\rangle \otimes \langle\lambda_i|\,, \qquad i = 0,\ldots,n_L-1\,. \tag{3.9}$$

The gauge invariant current is then given by:

$$j_L^{\bar{i}i} = q|\bar{\lambda}_i\rangle\langle\lambda_i|\,. \tag{3.10}$$

Under translations this current transforms non trivially, in fact under a spatial translation by an amount Δ_i, with $\Delta = \Delta_1 + i\Delta_2$,

$$j_L^{\bar{i}i} = q|\bar{\lambda}_i\rangle\langle\lambda_i| \longrightarrow e^{-c\bar{\Delta}+c^\dagger\Delta} j_L^{\bar{i}i} e^{-c\bar{\Delta}+c^\dagger\Delta} = q|\bar{\lambda}_i+\bar{\Delta}\rangle\langle\lambda_i+\Delta|\,. \tag{3.11}$$

Therefore we see that the λ_i can be interpreted as the positions of the vortices. If we restore θ we would find that the positions x_i^m of the vortices are gven by $x_i^m = \theta^{\mu\nu}\lambda_{i\nu}$ (D and thus λ_i have the dimensions of momenta.) If we use (2.9) to write the current as an ordianry function we find that

$$j_L^{\bar{i}i}(x) \propto \exp\left[\frac{-(x-x_i)^2}{2\theta}\right]\,, \tag{3.12}$$

curresponding to a current density that is a gaussian localized at position x_i, of width proportional to $\sqrt{\theta}$.

3.3 Probing the vortices with Wilson loops

The above discussion makes it clear that the moduli λ_i correspond to the position of the vortices. The positions of the vortices can also be detected using the Wilson loop operators of definite momenta. Let us evaluate the operators (2.33) that correspond to a Wilson loop of definite momentum in the background of the q-vortex solution (3.6) . We reintroduce θ in this section. The trace separates as

$$\mathcal{W}(\vec{l}) = \mathrm{Tr}\prod\nolimits_j \exp i\mathcal{D}(l_j) = \sum_{i=0}^{q-1} e^{i\vec{\ell}\cdot\vec{\lambda}_i} + \mathrm{Tr}\prod\nolimits_j \exp\frac{i}{\sqrt{2\theta}}\left[S_q(\bar{l}_j c^\dagger + l_j c){S_q}^\dagger\right]\,, \tag{3.13}$$

where $\vec{\ell} = \sum_j \vec{l}_j$. The second term in (3.13) is actually identical to the Wilson loop (2.33) in the vacuum, since using ${S_q}^\dagger S_q = 1$, $S_q {S_q}^\dagger = 1 - P_q$, we have

$$\begin{aligned}
\mathrm{Tr} \prod\nolimits_j \exp \frac{i}{\sqrt{2\theta}} \left[S_q (\bar{l}_j c^\dagger + l_j c) {S_q}^\dagger \right] &= \mathrm{Tr}\, S_q \left[\prod\nolimits_j \exp \frac{i}{\sqrt{2\theta}} (\bar{l}_j c^\dagger + l_j c) \right] {S_q}^\dagger \\
&= \mathrm{Tr} \prod\nolimits_j \exp \frac{i}{\sqrt{2\theta}} (\bar{l}_j c^\dagger + l_j c) \\
&= \frac{\delta^2(\ell)}{2\theta} \exp \left[\frac{i\mathcal{A}}{2\theta} \right] . \qquad (3.14)
\end{aligned}$$

Where $\mathcal{A}$ is the (oriented) area spanned by the loop (delta function makes the loop close), $\mathcal{A} = \frac{1}{2i} \sum_{i<j} \left(l_i \bar{l}_j - l_j \bar{l}_i \right) = \frac{1}{2} \oint x_1 dx_2 - x_2 dx_1$.

Thus, in the leading semi-classical approximation

$$\langle \mathcal{W}(\vec{l}) \rangle = \sum_{i=0}^{q-1} e^{i\vec{\ell}\cdot\vec{\lambda}_i} + \frac{\delta^2(\ell)}{2\theta} \exp \left[\frac{i\mathcal{A}}{2\theta} \right] .$$

Consequently $\mathcal{W}(\vec{l})$, for non-vanishing momentum $p_\mu = \theta^{-1}_{\mu\nu} \ell^\nu$, couples to the vortices which behave as local, pointlike sources at the positions $x^\mu = \theta^{\mu\nu} \lambda_\nu$. By measuring $\mathcal{W}(\vec{l})$, for different $\vec{l}$'s (note that as we change $\vec{l}$ we change the loop in $\mathcal{W}(\vec{l})$), we can determine the positions of the vortices with arbitrary accuracy.

Alternatively we can use the non-commutative analogue of the local energy density, as constructed in [16], to explore the multi-vortex configuration. This operator carries momentum p and is given by

$$\mathcal{E}(p) = \mathrm{Tr} \left[\exp \mathcal{D}(\theta^{\mu\nu} p_\nu)\, F^2 \right] , \qquad (3.15)$$

where a straight Wilson line has been inserted. In the background of the multi-vortex solution, $F^2 = P_q$ and thus

$$\mathcal{E}(p) = \sum_{i=0}^{q-1} \frac{2\pi}{g^2\theta} e^{ip_\nu x_i^\nu} ,$$

as if we have a vortex of energy $\frac{2\pi}{g^2\theta}$ localized at the positions, $x_i^\mu = \theta^{\mu\nu} \lambda_{\nu,i}$. The Fourier transform of $\mathcal{E}(p)$ would give an operator with local (delta-function) support at these positions, however this is not the Fourier transform of a given operator since as p, and thus l, changes so does the length of the Wilson line.

There appears to be a contradiction between the behavior of the current densities that indicate that the vortices are spread out in position space (over a size $\sqrt{\theta}$) and the behavior of the Wilson loop or the gauge invariant energy density, which indicate pointlike structure. We believe that the current densities are more reliable, physical probes of the vortices. This is because the current densities have a cannonical normalization (the Ψ's are normalized eigenvectors), whereas the normalization of $\mathcal{E}(p)$

or $\mathcal{W}(\vec{l})$ is somewhat arbitrary. (In [16] this problem was dealt with by considering ratios of 3 and 2-point functions of these operators). If we were to multiply the loop operators by, say $\exp[-p^2\theta]$, this would not change the total energy or the closed Wilson loop and would produce the expected form factors.

3.4 Translating the vortices

We should also be able to see that λ_i correspond to the positions of the vortices by performing a translation on the background fields. As we discussed before, the translations of D and $D^\dagger$ in the non-commutative plane are generated by the gauge transformations and constant shifts of A. Thus, up to gauge tranformation, translations by an amount x^μ simply correspond to shifts of D ($\bar{D}$) by x/θ ($\bar{x}/\theta$) — which has the effect of shifting the λ_ν by $\theta_{\nu\mu}x^\mu$. Thus, as derived above, the position of the i-th vortex is $x_i^\mu = \theta^{\mu\nu}\lambda_{\nu,i}$.

3.5 Stability

In the commutative two-dimensional theory on an infinite plane one cannot have stable localized droplets of magnetic flux. A vortex of quantized flux, Q, can be constructed by having the field equal to Q/A, in a region of area A. The energy will be proportional to $(Q^2/A^2)A = Q^2/A$. This simple energy consideration implies that a drop of flux will immediately spread out to fill all of the space, and will have vanishing field strength in any finite region. In the non-commutative setup the vortices that we have constructed are classical solutions. But here too they are not stable — rather they are metastable and will decay and spread out if perturbed.

First of all, notice that the solutions with different magnetic charge can be continuously connected in field space. For example, let us take the charge one solution $D = -S_1 c^\dagger {S_1}^\dagger$ and connect it to the vacuum solution $-c^\dagger$ by a path:

$$D_\tau = -\left(\tau S_1 c^\dagger {S_1}^\dagger + (1-\tau)c^\dagger\right), \qquad \tau \in [0,1]. \tag{3.16}$$

For every value of τ the gauge field defined by (3.16) is well defined,

$$A_\tau = -\tau S_1[c^\dagger, {S_1}^\dagger]. \tag{3.17}$$

Note that this would not have been the case if we decided, say, to connect in the same vein the Dirac monopole and the trivial gauge field on a two-dimensional sphere. Moreover, one can compute the flux of this gauge field, and its energy:

$$\begin{aligned}\mathrm{Tr}\, F_\tau = \tau, \qquad \mathrm{Tr} F_\tau^2 &= \tau^2(2-\tau)^2 + 4\tau^2(1-\tau)^2 \sum_{m=1}^{\infty}\left(\frac{2\sqrt{m}}{\sqrt{m+1}+\sqrt{m-1}} - 1\right)^2 \\ &= \tau^2\left[(2-\tau)^2 + 4a(1-\tau)^2\right], \qquad a \approx 0.173153.\end{aligned} \tag{3.18}$$

We see that the energy is finite for any τ, that at $\tau = 0$ it has a minimum, and at $\tau = 1$ it has a local maximum, while it monotonically increases in between. We also see, that at $\tau = 1$ there is a negative mode in the expansion of the energy around the solution D_1.

Indeed, it can be verified, along the lines of [11] that the spectrum of fluctuations around the d-vortex solution contains a tachyonic mode. This mode was absent in the (essentially the same) analysis of the fluctuations of the d-fluxons, presented in [11] due to the contribution of the Higgs field, which is absent in the solution (3.6). To see this expand about the vortex solution,

$$D = D_0 + A\,, \qquad \bar{D} = \bar{D}_0 + \bar{A}\,; \qquad \bar{D}_0 = \lambda_q - S_q c^\dagger S_q{}^\dagger\,.$$

The quadratic part of the action becomes

$$\mathcal{L}_2 = \int dt \mathrm{Tr}\left\{-2\partial_t A \partial_t \bar{A} - 2P_q[\bar{A}, A] - ([\bar{D}_0, A] - [D_0, \bar{A}])^2\right\}. \tag{3.19}$$

As in [11] the fluctuating field A can be decomposed into four pieces, which would correspond to modes due to $0-0$, $0-2$, $2-0$, $2-2$ strings, respectively. The $0-0$ modes, for example, correspond to modes that lie in the d-dimensional subspace V_q, namely

$$A = \sum_{i,k=0}^{d-1} a_{ik}|i\rangle\langle k|\,,$$

These modes contribute to $\mathcal{L}_2$ the terms:

$$\int dt \sum_{i,k=0}^{d-1}\left\{2|\partial_t a_{ik}|^2 - |a_{ik}|^2(\lambda_i - \lambda_j)^2\right\}, \tag{3.20}$$

This is the quadratic piece of the $(0+1)$-dimensional field theory of the D_0-branes, at positions λ_i. The diagonal components, a_{ii}, are the translation zero modes; whereas the off-diagonal terms acquire a (higgs) mass for separated vortices.

The unstable mode is actually in the off diagonal sector of the Hilbert space. Consider the single vortex, $d = 1$. In that case it is easy to verify that the mode: $A = |1\rangle\langle 0|b(t)$, satisfies the equation of motion: $\ddot{b} = b$, and is tachyonic.

A nice picture of the fluctuation spectrum can also be seen by restricting to a subspace of $\mathcal{F}$, which consists of the gauge fields which have the form:

$$D = -f(N)c^\dagger\,, \qquad N = c^\dagger c\,.$$

Such gauge fields form an invariant subspace with respect to the time evolution generated by the gauge theory hamiltonian. Indeed, the field strength is diagonal. The potential energy becomes a functional on $f(N)$:

$$V = \sum_{n=0}^{\infty}\left(n|f(n)|^2 - (n+1)|f(n+1)|^2 + 1\right)^2. \tag{3.21}$$

Introduce $x_n = n|f(n)|^2$. The d-vortex solution has:

$$x_n = 0\,, \qquad n \le d\,; \qquad x_n = n - d\,, \qquad n > d\,, \tag{3.22}$$

Expanding V around this solution we get:

$$V = d - 2x_d^2 + O(x^4)\,, \tag{3.23}$$

and we can identify $f(d)$ as the tachyon mode. For $d = 1$ this mode coincides with the mode described above.

3.6 Unstable solutions and exact path integrals

The d-vortex solutions are similar to the unstable monopole solutions of the Yang-Mills equations on a two-dimensional sphere, for the gauge group SU(N). Consider, for simplicity, the gauge group SU(2). All the solutions are classified by a non-negative integer d, and have the form:

$$A = d \begin{pmatrix} A_{\mathrm{Dir}} & 0 \\ 0 & -A_{\mathrm{Dir}} \end{pmatrix},$$

where A_{Dir} is a constant curvature U(1) gauge field on a two-sphere (which can be obtained by restricting the Dirac monopole to the sphere). The action of such a solution is:

$$S_d = \frac{2\pi^2 d^2}{g^2 A}\,.$$

The partition function, as a function of the area A of the sphere and the coupling constant g, is given by [17]:

$$Z = \sum_{n=1}^{\infty} n^2 e^{-\frac{1}{2}g^2 A n^2} = \frac{\sqrt{2\pi}}{g^3 A^{3/2}} + \sum_{d=1}^{\infty} \frac{\sqrt{8\pi}}{g^3 A^{3/2}} \left(1 - \frac{4\pi^2 d^2}{g^2 A}\right) e^{-S_d} \tag{3.24}$$

that is, for this theory, the semi-classical approximation, together with a finite number of quantum fluctuations, is exact, provided one sums over all critical points of the action.

Let us now discuss the partition function of the two-dimensional euclidean noncommutative gauge theory. It is given by:

$$Z = \int \mathcal{D}D\mathcal{D}D^{\dagger} \exp - \left\{ \frac{\pi\theta}{2g^2} \operatorname{Tr}\left([D, \bar{D}] - 1\right)^2 \right\}. \tag{3.25}$$

We immediately see, that if we were to apply the WKB approximation to this theory, the partition function would diverge, for the unstable critical points given by (3.6) have moduli, given by the eigenvalues of λ_i, and integrating along these moduli would render partition function divergent. We propose to regularize this partition function by adding a gauge invariant term

$$\varepsilon \mathrm{Tr} D D^{\dagger}$$

to the action. This is the analogue of the infrared regularization provided by the area of the two-sphere in the commutative example.

The related model

$$Z_\varepsilon = \sum_N e^{\mu N} \int_{\mathrm{Mat}_{N\times N}} \frac{\mathcal{D}D\mathcal{D}D^\dagger}{\mathrm{Vol}(\mathrm{U}(N))} \exp - \left\{ \frac{\pi\theta}{2g^2} \,\mathrm{Tr}\left([D,\bar{D}]-1\right)^2 + \varepsilon \mathrm{Tr} DD^\dagger \right\}, \quad (3.26)$$

is extremely rich and can be solved exactly [18]. Whether this solution can lead to an exact solution of $2d$ NC Yang-Mills theory is a question that deserves further study.

3.7 Supersymmetric solutions

In the supersymmetric gauge theory, in addition to the gauge fields we have fermions and scalars. The above analysis is easily extended to include these fields. Let us discuss the case of the maximally supersymmetric theory.

In addition to the gauge fields, entering $D, \bar{D}$ we have a collection of 7 (for the $(2+1)$-dimensional theory) scalar fields Φ_a, corresponding to the 7 transverse directions to D3-branes. These transform in the adjoint representation and thus are to be identified with hermitean operators Φ^a in the Fock space $\mathcal{H}$. The full bosonic part of the action is given by (in the $A_t = 0$ gauge):

$$S = \frac{2\pi\theta}{4g^2} \int \mathrm{d}t \,\mathrm{Tr} \left[4\dot{D}\dot{\bar{D}} + \sum_a \dot{\Phi}^a\dot{\Phi}^a + \right. \quad (3.27)$$

$$\left. + \sum_a [D,\Phi_a][\bar{D},\Phi_a] + \sum_{a\neq b} [\Phi_a,\Phi_b]^2 + 4\left([D,\bar{D}] - \frac{1}{2\theta}\right)^2 \right].$$

It is convenient to unify the Higgs fields and the gauge fields into a single **10**-plet of (anti-hermitean) operators D_A, $A = 0,\ldots,9$, acting in some Hilbert space $\mathbf{H}$. The gauge theory is then described by the IKKT action [19]:

$$S = -\frac{1}{4g^2} \sum_{A<B} \mathrm{Tr}_{\mathbf{H}} \left([D_A,D_B] + i\theta_{AB}\right)^2 + \text{fermions}\,. \quad (3.28)$$

The equations of motion following from (3.28) are (setting all fermions to zero):

$$\sum_A [D_A,[D_A,D_B]] = 0\,. \quad (3.29)$$

A special class of solutions to (3.29) are provided by the field configurations that obey a stronger condition than (3.29):

$$[D_A,[D_B,D_C]] = 0\ ,$$

for any A,B,C. Repeating the arguments in (3.2), (3.3) and (3.4), we arrive at the following generic classification of the solutions with the finite p-tension:

$$[D_A,D_B] = i\left(-\theta_{AB} - f_{AB}P_{AB}\right), \quad (3.30)$$

where $P_{AB} = P_{BA} = P_{AB}^{\dagger}$ are projectors in $\mathbf{H}$, and f_{AB} are c-numbers . The collection of the projectors P_{AB} must have the following properties:

$$[D_C, P_{AB}] = 0\,,$$
$$\sum_{A<B} f_{AB}^2 \, \mathrm{Tr}_{\mathbf{H}} P_{AB}^2 \sim V_p\,,$$

where V_p is the volume of the p-brane. The operators D_A generate a certain algebra, whose *spectrum* coincides with the worldvolume of the configuration of the D-branes that the solution (3.30) corresponds to.

For example, the vacuum solution corresponding to a single flat Dp-brane, extended in the directions $0, 1, \ldots, p$, without any B-field (i.e. $\theta_{AB} = 0$) is:

$$D_\mu = \partial_\mu\,, \qquad \mu = 0, \ldots, p; \qquad D_A = ix^A\,, \qquad A > p\,,$$

where x^A are the coordinates of the brane in the transverse space. The Hilbert space is in this case the space $\mathbf{H} = S(\mathbb{R}^{1,p})$ of smooth functions of $(x^0, \ldots, x^p)$. If we were to consider N parallel flat branes then $\mathbf{H} = S(\mathbb{R}^{1,p} \times \{1, \ldots, N\})$, and the solution would involve $9 - p$ commuting $N \times N$ matrices Φ^A, $A = p, \ldots, 9$. The spectrum consists in this case of N copies of the space $\mathbb{R}^{1,p}$ (the latter emerges as the set of eigenvalues of the operators ∂_μ).

In general the solution (3.30) represents a collection of Dp'-branes of various dimensionalities p'. As usual, one can read off the D-brane charges from the Chern character:

$$\mathrm{Tr}_{\mathbf{H}} \exp \frac{1}{2\pi i} i f_{AB} dx^A \wedge dx^B P_{AB}\,.$$

The solution representing a D2-brane and a collection of q D0-branes located at various points in the nine-dimensional space, with B-field being in the 12 directions, is given by:

$$\begin{aligned} D_0 &= \partial_0\,, \\ D_A &= \lambda_A + i\theta_{AB} S\, x^B\, S^{\dagger}\,, \qquad A = 1, \ldots, 9\,, \end{aligned} \tag{3.31}$$

with λ_A being a diagonal matrix in the N-dimensional subspace V of the Hilbert space $\mathbf{H} = S(\mathbb{R}^{1,0}) \otimes \mathcal{H}$. The projectors P_{AB} are given by:

$$P_{12} = P_{21} = \int dt \sum_{l=0}^{N-1} |l\rangle\langle l|\,,$$

with the rest vanishing.

Another interesting static solution, representing branes of different dimensions, is given by:

$$\begin{aligned} D_0 &= \partial_0\,, \qquad D_3 = \partial_3\,, \\ D_A &= \lambda_A + \mu_A x^3 + i\theta_{AB} S\, x^B\, S^{\dagger}\,, \qquad A = 1, 2, 4 \ldots, 9 \end{aligned} \tag{3.32}$$

with λ_A, μ_A being the diagonal matrices in the N-dimensional subspace V of the Hilbert space $\mathbf{H} = S(\mathbb{R}^{1,1}) \otimes \mathcal{H}$. In this case we see a D3-brane, extended in the $0, 1, 2, 3$ directions, with $1, 2$ directions being non commutative, and a collection of N D1-strings, forming various angles with the D3-brane. This solution has $16N$ moduli (of which $\sim \mathrm{Tr}\,\lambda_1, \mathrm{Tr}\,\lambda_2$ can be eliminated by a gauge transformation generating translations).

The stability of these solutions is analyzed by diagonalizing the operator of quadratic fluctuations about the solution:

$$\delta^2 S\, a_A = -\sum_B [D_B, [D_B, a_A]] + 2[[D_B, D_A], a_B]\,, \tag{3.33}$$

where $\delta D_A = a_A$ and the gauge condition

$$[D_A, a_A] = 0\,, \tag{3.34}$$

has been imposed.

In the case of 0-2, 1-3 systems the only projectors involved were the projectors P onto the N-dimensional subspace V of the Hilbert space $\mathbf{H}$. Every operator $\mathcal{O}$ in $\mathbf{H}$ is canonically decomposed into four components:

$$\begin{aligned} \mathcal{O} &= \mathcal{O}^{VV} + \mathcal{O}^{VH} + \mathcal{O}^{HV} + \mathcal{O}^{HH}\,, \\ \mathcal{O}^{VV} &= P\mathcal{O}P\,, \\ \mathcal{O}^{VH} &= P\mathcal{O}(1-P)\,, \qquad \mathcal{O}^{HV} = (1-P)\mathcal{O}P\,, \\ \mathcal{O}^{HH} &= (1-P)\mathcal{O}(1-P)\,. \end{aligned} \tag{3.35}$$

The fluctuation modes $\mathcal{O}^{HH}$ are identical to the massless fields propagating on a single D2- (D3-) brane. The modes $\mathcal{O}^{VV}$ coincide with the fields of a matrix model describing q D0-branes or with those of rank q matrix strings.

The modes $\mathcal{O}^{VH}, \mathcal{O}^{HV}$ are the interesting ones: they contain tachyons (for unstable configurations) corresponding to the decay of the lower-dimensional branes inside of the D2 (D3). They also sometimes contain extra massless modes, responsible for breaking of D1 strings into two semi-infinite strings, ending on D3-brane [11].

3.8 Plane wave solutions

In the case of 1-3 system one expects tofind exact plane waves propagating along the string worldsheet.

Indeed, the following exace solution generalizes (3.32) and describes a collection of q D1 strings with plane wave excitations propagating along them:

$$\begin{aligned} D_0 &= \partial_0, \qquad D_3 = \partial_3\,, \\ D &= f - S c^\dagger S^\dagger\,, \end{aligned}$$

$$
\begin{aligned}
D_a &= i f_a \,, \\
f &= \sum_{l=0}^{q-1} f_l(x^0, x^3)|l\rangle\langle l| \,, \qquad f_a = \sum_{l=0}^{q-1} f_{a,l}(x^0, x^3)|l\rangle\langle l| \,, \\
\left(\partial_0^2 - \partial_3^2\right) f &= 0 \,, \qquad a = 4, \dots, 9 \,, \\
\left(\partial_0^2 - \partial_3^2\right) f_a &= 0 \,.
\end{aligned} \tag{3.36}
$$

These solutions are not BPS and might be unstable to radiating 3-3 strings into the bulk. The stability of these solutions will be analyzed elsewhere.

3.9 Static strings

The equations of motionthat follow from (3.27) (setting all the fermions to zero), for static fields are:

$$
\begin{aligned}
[D, [\bar{D}, \Phi_a]] + \sum_{b \neq a} [\Phi_b, [\Phi_b, \Phi_a]] &= 0 \,, \\
[\Phi_a, [\bar{D}, \Phi_a]] + [\bar{D}, F] &= 0 \,, \\
F &= [D, \bar{D}] - 1 \,.
\end{aligned} \tag{3.37}
$$

These equations are easily solved. Choose D and $\bar{D}$ to be as before, namely as in (3.6). Then the equations for the scalars reduce to:

$$
\begin{aligned}
\sum_{b \neq a} [\Phi_b, [\Phi_b, \Phi_a]] &= 0 \,, \\
[\Phi_a, [\bar{D}, \Phi_a]] = [\Phi_a, [D, \Phi_a]] &= 0 \,.
\end{aligned} \tag{3.38}
$$

By choosing

$$
\Phi_a = \underline{\lambda}_q^a = \sum_{i=0}^{q-1} \lambda_i^a |i\rangle\langle i| \,,
$$

we clearly obey all of these equations. In addition we can, of course shift all the Φ_a by multiples of the identity.

These solutions can be lifted to higher dimensions, where more parameters appear. For example, in the (3+1)-dimensional theory, on a non-commutative space with coordinates t, x^3 commuting and x^1, x^2 non commuting, the operator Φ_7 can be regarded as a gauge field in the 3 direction:

$$
\Phi_7 = \partial_3 + A_3 \,,
$$

and one can solve (3.38) by setting

$$
\begin{aligned}
A_3 &= 0 \,, \\
\Phi_a &= \underline{\lambda}_q^a + \underline{\mu}_q^a \, x^3 \,, \qquad a = 1, \dots, 6 \,, \\
D &= \lambda_q + \mu_q \, x^3 - S^\dagger c^\dagger S \,,
\end{aligned} \tag{3.39}
$$

where $\underline{\mu}_q^a$ are a set of diagonal $q \times q$ matrices in V_q. This solution describes a collection of q D1-strings forming different angles with D3-brane, set by the eigenvalues of the operators $\underline{\mu}_q^a, \mu_q$. In the following we assume, for simplicity, that $\mu_q = 0$.

Let us define an operator $|\mu_q|$ as the diagonal $q \times q$ matrix:

$$|\mu_q| = \sqrt{\sum_{a=1}^{6} \left(\mu_q^a\right)^2}\,.$$

If all the D1-strings are parallel to each other and form a critical angle with the D3-brane then the solution (3.39) describes a general BPS q-fluxon of [11], with all moduli turned on (this solution was announced in [11]). In particular it solves Bogomolny equation:

$$B_i + D_i \Phi = 0\,, \qquad i = 1, 2, 3\,; \qquad \Phi^a = \lambda_q^a + n^a \Phi\,,$$

where $\sum_a (n^a)^2 = 1$, which means that $\mu_q^a = 2n^a \cdot \mathrm{I}_q$. This solution is stable.

However, for generic values of μ_d the solutions have negative modes. We shall now analyze these instabilities. As in [11] the fluctuations around the solution (3.39) split into 1-1, 1-3, 3-1, and 3-3 sectors according to the decomposition of an arbitrary operator $\mathcal{O}$ acting in $\mathcal{H}$:

$$\mathcal{O} = P_q \mathcal{O} P_q + P_q \mathcal{O}(1 - P_q) + (1 - P_q)\mathcal{O} P_q + (1 - P_q)\mathcal{O}(1 - P_q)\,. \tag{3.40}$$

Let us denote by a_μ the fluctuations of A_μ and by φ^a the fluctuations of Φ^a, and by $X = \frac{1}{2}(a_1 + ia_2)$ the fluctuations of D:

$$X = \delta D\,, \qquad a_3 = \delta A_3 = -a_3^\dagger\,, \qquad \varphi^a = \delta\Phi^a = \varphi^{a\dagger}\,. \tag{3.41}$$

We can split the fluctuations of the scalars into those that are transverse to the strings and into longitudinal fluctuations:

$$\varphi^a = \bar{\varphi}^a + ({\mu^a}_q \zeta + \zeta \mu_q^a)\,,$$

where $\sum_a \mu^a \bar{\varphi}^a = \sum_a \bar{\varphi}^a \mu^a = 0$. The operator ζ can belong to the 1-1, 1-3, or the 3-1 sectors. Define the operator Y to be:

$$Y = a_3 + |\mu_q|\zeta + \zeta|\mu_q|\,.$$

The 3-3 modes, corresponding to strings attached to the D3-brane, are identical to those in [11]. The instabilities appear in the 1-3 and 3-1 sectors. They have the following equations of motion (cf. with [11, eq. (4.7)]. Note that we have imposed

Lorentz gauge conditions on the fluctuations):

$$
\begin{aligned}
1-3: \qquad & \left(\partial_t^2 - \partial_3^2 + 2(2\hat{n}+1) + \sum_a (\lambda_q^a + \mu_q^a x_3)^2 + 4\right) X = 0\,, \\
& \left(\partial_t^2 - \partial_3^2 + 2(2\hat{n}+1) + \sum_a (\lambda_q^a + \mu_q^a x_3)^2 + 2|\mu_q|\right) Y = 0\,, \\
& \left(\partial_t^2 - \partial_3^2 + 2(2\hat{n}+1) + \sum_a (\lambda_q^a + \mu_q^a x_3)^2\right) \bar{\varphi}_b = 0\,. \\
3-1: \qquad & \left(\partial_t^2 - \partial_3^2\right) X + X\left(2(2\hat{n}+1) + \sum_a (\lambda_q^a + \mu_q^a x_3)^2 - 4\right) = 0\,, \\
& \left(\partial_t^2 - \partial_3^2\right) Y + Y\left(2(2\hat{n}+1) + \sum_a (\lambda_q^a + \mu_q^a x_3)^2 - 2|\mu_q|\right) Y = 0\,, \\
& \left(\partial_t^2 - \partial_3^2\right) \bar{\varphi}_b + \bar{\varphi}_b\left(2(2\hat{n}+1) + \sum_a (\lambda_q^a + \mu_q^a x_3)^2\right) = 0\,.
\end{aligned} \tag{3.42}
$$

where

$$
\begin{aligned}
\hat{n}|i\rangle\langle\psi| &= |i\rangle\langle\psi|\left(-q + (c^\dagger - \bar{\lambda}_i)(c - \lambda_i)\right), \qquad i = 0, \ldots, q-1\,, \\
\hat{n}^\dagger &= \hat{n}\,.
\end{aligned}
$$

The resulting spectrum of the 1-3, 3-1 fluctuations is:

$$
\begin{aligned}
& \partial_t^2 = -\omega^2\,, \qquad \omega_i^2 = m_i + \varepsilon_i\,, \\
& m_i = \left(\sum_a (\lambda_i^a)^2\right) - \left(\frac{1}{|\mu_q|_i} \sum_a \lambda_i^a \mu_i^a\right)^2, \\
1-3: \qquad & X:\ \varepsilon_i = |\mu_q|_i(2m+1) + 2(2n+3)\,, \\
& Y:\ \varepsilon_i = |\mu_q|_i(2m+3) + 2(2n+1)\,, \\
& \bar{\varphi}_a:\ \varepsilon_i = |\mu_q|_i(2m+1) + 2(2n+1)\,, \\
3-1: \qquad & X:\ \varepsilon_i = |\mu_q|_i(2m+1) + 2(2n-1)\,, \\
& Y:\ \varepsilon_i = |\mu_q|_i(2m-1) + 2(2n+1)\,, \\
& \bar{\varphi}_a:\ \varepsilon_i = |\mu_q|_i(2m+1) + 2(2n+1)\,,
\end{aligned} \tag{3.43}
$$

with $m, n \geq 0$.

The quantity m_i determines whether the i-th D1-string pierces the D3-brane. In a 9-dimensional space a line and a 3-plane do not intersect in general. The shortest distance between them is $\sqrt{m_i} \geq 0$. Thus we see that if for some $i = 0, \ldots, q-1$ $||\mu_q|_i - 2| > m_i$ then there is a tachyonic mode, either for X (if $|\mu_q|_i < 2$), or for Y (if $|\mu_q|_i > 2$).

4. U(2) monopoles

So far all of our discussion has been devoted solely to the U(1) non-commutative gauge theories that arise in the Seiberg-Witten $\alpha' \to 0$ limit of Dp-brane theories with a B-field turned on. The solitons we constructed were localized in the non-commutative directions, but generically occupied all of the commutative space, corresponding to (semi)infinite D$(p-2)$-branes, immersed in a Dp-brane, or piercing it. We now describe solitons which, although they have finite extent in the commutative directions, are nevertheless localized and look like codimension three objects when viewed from far away. The simplest such object is the monopole in the non-commutative U(2) gauge theory, i.e. the theory on a stack of two separated D3-branes in the Seiberg-Witten limit [4].

We are interested in the U(2) gauge theory on the non-commutative three-dimensional space. Let $H \approx \mathbb{C}^2$ be the Chan-Paton space, i.e. the fundamental representation for the commutative limit of the gauge group, and let e_0, e_1 denote an orthonormal basis in H. The non-commutative version of the fundamental representation is infinite dimensional, isomorphic to $H \otimes \mathcal{H}$. That is, the U(2) matter fields Ψ belong to the space $\mathcal{H} \otimes \mathrm{Fun}(x^3) \otimes (\mathcal{H} \otimes H)$, where the first two factors make it a representation of the algebra $\mathcal{A}_\theta$ of non-commutative functions on $\mathbb{R}^3$, while the second two factors make it a representation of the U(2) non-commutative gauge group. Actually, the latter is isomorphic to the group of (x^3-dependent) unitary operators in the Hilbert space $\mathcal{H} \otimes H$. Now, the Hilbert space $\mathcal{H} \otimes H$ is isomorphic to $\mathcal{H}$ itself:

$$|n\rangle \otimes e_\alpha \longleftrightarrow |2n+\alpha\rangle \,. \tag{4.1}$$

We wish to solve Bogomolny equations:

$$[D_i, \Phi] = \frac{i}{2}\varepsilon_{ijk}[D_j, D_k] - \delta_{i3}\theta \,, \tag{4.2}$$

where Φ and D_i, $i = 1,2,3$ are the operators in $\mathcal{H} \otimes H$ which have a non-trivial magnetic charge:

$$Q_m = \int dx^3 \mathrm{Tr}_{\mathcal{H}} \partial_i \left(\mathrm{Tr}_H \Phi B_i\right), \tag{4.3}$$

where

$$B_i = \frac{i}{2}\varepsilon_{ijk}[D_j, D_k] - \delta_{i3}\theta \,,$$

and the Higgs field Φ approaches

$$\begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix} \otimes \mathrm{I}_{\mathcal{H}}$$

as $x_3^2 + 2\theta c^\dagger c \to \infty$.

4.1 Nahm's equations

It was found in the study of commutative gauge theories that the BPS solutions of gauge theory can be found via a sort of Fourier transform, or reciprocity transformation [20]. In the instanton case the four-dimensional anti-self-duality equations are mapped to matrix ADHM equations [20]. In the monopole case the three-dimensional equations go over to one-dimensional matrix differential equations — Nahm's equations [21]. As explained, e.g. in [22], Nahm's equations are the BPS equations for D1-strings suspended between D3-branes. (At the same time the ADHM equations are analogous equations for D(−1)-branes dissolved inside D3-branes). In [10, 23] the non-commutative version of Nahm's equations was derived. They have the form:

$$\partial_z T_i = \frac{i}{2}\varepsilon_{ijk}[T_j, T_k] - \theta\delta_{i3}\,, \tag{4.4}$$

where in the case of the U(2) gauge theory the matrices T_i have size $k \times k$, with k being the monopole charge, the parameter z takes values in the interval $I = [a_-, a_+]$, and T_l have first order poles at the ends of I with residues forming a k-dimensional irreducible representation of U(2).

For $k = 1$ this means that the matrices must be regular everywhere, which in turn yields the unique solution:

$$T_i(z) = \theta\delta_{i3} z + \kappa_i\,, \tag{4.5}$$

where κ_i are arbitrary constants. This solution represents a tilted D1-string suspended between two D3-branes, located at $z = a_-$ and $z = a_+$, respectively.

The next step is to find a two-component spinor vector-function

$$\Psi(z,\vec{x}) = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix},$$

which obeys the equation:

$$\partial_z \Psi = i\sigma_i \left(T_i(z) + x_i\right)\Psi\,, \tag{4.6}$$

where σ_i are the Pauli matrices and x_i are the spatial coordinates, i.e. the generators of $\mathcal{A}_\theta$. The fundamental solution to (4.6) is a $k \times 2$ spinor-valued matrix (both Ψ_+ and Ψ_- are $k \times 2$ matrices whose entries belong to $\mathcal{A}_\theta$). The solution to (4.6) is defined up to right multiplication by an element of $\mathrm{Mat}_2(\mathcal{A}_\theta) \approx \mathcal{A}_\theta \otimes \mathrm{End}(H)$. Among these elements the unitary elements (i.e. the ones which solve the equation $uu^\dagger = u^\dagger u = 1$) are considered to be the gauge transformations. In the commutative setup one normalizes Ψ as follows:

$$\int dz\, \Psi^\dagger \Psi = \mathrm{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.7}$$

Finally, given Ψ the solution for the gauge and Higgs fields is given explicitly by:

$$\Phi = \int dz\, z\, \Psi^\dagger \Psi\,,$$
$$A_i = \int dz\, \Psi^\dagger \partial_i \Psi\,. \tag{4.8}$$

For $k = 1$, by shifting x_i we can always set $\kappa_i = 0$.

4.2 Commutative case

We start with the commutative case, as it will be one of the limits to which our solution reduces as $\theta \to 0$. In the case $k = 1$ the analysis simplifies: $T_i = 0$, one can take $a_\pm = \pm a/2$ and

$$\Psi = \begin{pmatrix} (\partial_z + x_3)\, v \\ (x_1 + i x_2)\, v \end{pmatrix}, \qquad \partial_z^2 v = r^2 v\,, \qquad r^2 = \sum_i x_i^2\,. \tag{4.9}$$

The condition that Ψ is finite at both ends of the interval allows for two solutions of (4.9):

$$v = e^{\pm r z}\,,$$

which after imposing the normalization condition, (4.7), leads to:

$$\Psi = \frac{1}{\sqrt{2\sinh(ra)}} \begin{pmatrix} \sqrt{r + x_3}\, e^{rz} & -\sqrt{r - x_3}\, e^{-rz} \\ \frac{x_+}{\sqrt{r+x_3}} e^{rz} & \frac{x_-}{\sqrt{r-x_3}} e^{-rz} \end{pmatrix},$$

where $x_\pm = x_1 \pm i x_2$.

In particular,

$$\Phi = \frac{1}{2}\left(\frac{a}{\tanh(ra)} - \frac{1}{r}\right)\sigma_3\,.$$

4.3 Non-commutative case

In the case $k = 1$ we take: $T_{1,2} = 0, T_3 = \theta z$. Following [10] we introduce the operators:

$$b = \frac{1}{\sqrt{2\theta}}\left(\partial_z + x_3 + \theta z\right), \qquad b^\dagger = \frac{1}{\sqrt{2\theta}}\left(-\partial_z + x_3 + \theta z\right), \tag{4.10}$$

which obey $[b, b^\dagger] = 1$. We also introduce the superpotential W:

$$W = \frac{1}{2\theta}\left(x_3 + \theta z\right)^2, \tag{4.11}$$

whose importance arises from the formulae:

$$b = \frac{1}{\sqrt{2\theta}} e^{-W} \partial_z e^{W}\,, \qquad b^\dagger = -\frac{1}{\sqrt{2\theta}} e^{W} \partial_z e^{-W}\,. \tag{4.12}$$

It is convenient to choose units, where $2\theta = 1$.

Equations (4.6) then take the form:

$$\begin{aligned} b^\dagger \Psi_+ + c\Psi_- &= 0\,, \\ -c^\dagger \Psi_+ + b\Psi_- &= 0\,, \end{aligned} \tag{4.13}$$

where $\Psi_\pm(z) \in \mathcal{A}_\theta$. It is convenient to solve first the equation

$$\begin{aligned} b^\dagger \epsilon_+ + c\epsilon_- &= 0\,, \\ -c^\dagger \epsilon_+ + b\epsilon_- &= 0\,, \end{aligned} \tag{4.14}$$

with $\epsilon_\pm(z) \in \mathcal{H}$. The latter has the following solutions:

$$\begin{aligned} \varepsilon^\alpha &= \begin{pmatrix} \epsilon_+^\alpha \\ \epsilon_-^\alpha \end{pmatrix}, \qquad \alpha = 0, 1\,, \\ \varepsilon_0^0 &= \begin{pmatrix} 0 \\ \frac{1}{\sqrt{\zeta_0}} e^{-W} |0\rangle \end{pmatrix}, \qquad \zeta_0 = \int_{a_-}^{a_+} dz\, e^{-2W}\,, \qquad \varepsilon_0^1 = 0\,, \\ \varepsilon_n^\alpha &= \begin{pmatrix} b\, \beta_n^\alpha |n-1\rangle \\ \sqrt{n}\, \beta_n^\alpha |n\rangle \end{pmatrix}, \qquad n > 0\,. \end{aligned} \tag{4.15}$$

The functions β_n^0, β_n^1 solve

$$\left(b^\dagger\, b + n\right) \beta_n^\alpha = 0\,, \tag{4.16}$$

and are required to obey the following boundary conditions:

$$\begin{aligned} b\beta_n^1(a_+) &= 0\,, \qquad \beta_n^0(a_-) = 0\,, \\ \beta_n^1 b \beta_n^1(a_-) &= -1\,, \qquad \beta_n^0 b \beta_n^0(a_+) = 1\,. \end{aligned} \tag{4.17}$$

A solution to (4.13) is given by:

$$\Psi = \sum_{n\geq 0,\, \alpha=0,1} \varepsilon_n^\alpha \cdot \langle n-\alpha| \otimes e_\alpha^\dagger\,. \tag{4.18}$$

The conditions (4.17) together with (4.16) imply that:

$$\int dz\, \left(\varepsilon_n^\alpha\right)^\dagger \varepsilon_m^\gamma = \delta^{\alpha\gamma}\delta_{mn}\,,$$

which in turn yield (4.7). All other solutions to (4.13) are gauge equivalent to (4.18).

It is easy to generate the solutions to (4.16): first of all,

$$f_n(z) = b^{n-1}(e^W) = e^{W(z)} h_{n-1}(2x_3 + z)\,, \qquad h_k(u) = e^{-u^2/4} \frac{d^k}{du^k} e^{u^2/4}$$

is a solution. Then

$$\hat{f}_n = f_n(z) \int^z \frac{du}{f_n(u)^2}$$

is the second solution. Notice that, for k even, $h_k(u) > 0$ for all u and, for k odd, the only zero of $h_k(u)$ is at $u = 0$, and $h_k(u)/u > 0$ for all u. Therefore, $\hat{h}_k(z)$ is well defined for all z.

Consequently,

$$\begin{aligned}
\beta_n^0(z) &= \tilde{\nu}_n\, f_n(z) \int_{a_-}^{z} \frac{du}{f_n(u)^2}\,, \\
\beta_n^1 &= \nu_n \left(-\frac{1}{n f_{n+1}(z)} + f_n(z) \int_z^{a_+} \frac{du}{f_{n+1}(u)^2} \right),
\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}
\nu_n^{-2} &= \left(f_n(a_-) f_{n+1}(a_-) \int_{a_-}^{a_+} \frac{du}{f_{n+1}^2(u)} - \frac{1}{n} \right) \int_{a_-}^{a_+} \frac{du}{f_{n+1}^2(u)}\,, \\
\tilde{\nu}_n^{-2} &= \left(f_n(a_+) f_{n+1}(a_+) \int_{a_-}^{a_+} \frac{du}{f_n^2(u)} + 1 \right) \int_{a_-}^{a_+} \frac{du}{f_n^2(u)}\,.
\end{aligned} \tag{4.20}$$

(again, note that $\beta_n^\alpha(z)$ are regular at $z = -2x_3$).

We now are in position to calculate the components of the Higgs field and of the gauge field. We start with

$$\begin{aligned}
\Phi &= \int dz\, z\, \Psi^\dagger \Psi = \sum_{n \geq 0,\, \alpha,\gamma=0,1} \varphi_n^{\alpha\gamma} \cdot e_\alpha e_\gamma^\dagger \otimes |n-\alpha\rangle\langle n-\gamma|\,, \\
\text{where} \qquad \varphi_n^{\alpha\gamma} &= \int dz\, z \varepsilon_n^{\alpha,\dagger} \varepsilon_n^\gamma = -2x_3 \delta^{\alpha\gamma} + \int (b\beta_n^\alpha)(b+b^\dagger)(b\beta_n^\gamma) + n\beta_n^\alpha (b+b^\dagger) \beta_n^\gamma \\
&= -2x_3 \delta^{\alpha\gamma} + \left((b\beta_n^\alpha)(b\beta_n^\gamma) - n\beta_n^\alpha \beta_n^\beta \right) |_{a_-}^{a_+}\,.
\end{aligned} \tag{4.21}$$

The component A_3 of the gauge field vanishes, just as in the case of the U(1) solution of [10]:

$$\begin{aligned}
A_3 &= \int \Psi^\dagger \partial_3 \Psi = \int ((b\beta_n^\alpha)\partial_3(b\beta_n^\gamma) + n\beta_n^\alpha \partial_3 \beta_n^\gamma) \cdot e_\alpha e_\gamma^\dagger \otimes |n-\alpha\rangle\langle n-\gamma| \\
&= \frac{1}{2}\partial_3 \int \Psi^\dagger \Psi = 0\,.
\end{aligned} \tag{4.22}$$

The components A_1, A_2 can be read off from the expression for the operator D:

$$\begin{aligned}
D &= -\int dz\, \Psi^\dagger c^\dagger \Psi \\
&= \sum_{n \geq 0, \alpha,\gamma=0,1} D_n^{\alpha\gamma} \cdot\, e_\alpha e_\gamma^\dagger \otimes |n+1-\alpha\rangle\langle n-\gamma|\,, \\
\text{where} \qquad D_n^{\alpha\gamma} &= -\sqrt{n}\left(\beta_{n+1}^\alpha (b\beta_n^\gamma) \right) |_{a_-}^{a_+}\,.
\end{aligned} \tag{4.23}$$

The solution (4.21) and (4.23) has several interesting length scales involved (recall that our units above are such that $2\theta = 1$):

$$\theta |a_+ - a_-|\,, \qquad \sqrt{\theta}\,, \qquad \frac{1}{|a_+ - a_-|}\,.$$

By shifting x_3 we can always assume that $a_- = 0, a_+ = a > 0$.

4.4 Suspended D-string

In this section we set θ back to 1/2. As we discussed before the spectrum of the operators D_A, $A = 0, \ldots, 9$ determines the "shape" of the collection of D-branes the solution of the generalized IKKT model [1] corresponds to. To "see" the spatial structure of our solution let us concentrate on the $\langle 0|\Phi|0\rangle$ piece of the Higgs field, for it describes the profile of the D-branes at the core of the soliton. From (4.21) we see that

$$\langle 0|\Phi|0\rangle = \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \end{pmatrix},$$

where $\rho_+ = \varphi_0^{00}, \rho_- = \varphi_1^{11}$.

Let us look specifically at the component ρ_+ of the Higgs field:

$$\begin{aligned} \rho_+ &= -\frac{1}{2}\frac{\partial}{\partial x_3}\log\left(\int_0^a dp\, e^{-2x_3 p - \frac{1}{2}p^2}\right), \\ &= -2x_3 + \langle\langle p\rangle\rangle_{2x_3}^{a+2x_3}, \\ &= -2x_3 - 2\frac{e^{-(a+2x_3)^2/4} - e^{-(2x_3)^2/4}}{\gamma(a+2x_3) - \gamma(2x_3)}, \end{aligned} \tag{4.24}$$

where

$$\langle\langle\mathcal{O}\rangle\rangle_\alpha^\beta = \frac{\int_\alpha^\beta \mathcal{O} e^{-p^2/4}\, dp}{\int_\alpha^\beta e^{-p^2/4}\, dp}, \quad \gamma(z) = \int_0^{z/2} dp\, e^{-p^2/4}. \tag{4.25}$$

The $\langle\langle\cdots\rangle\rangle$ representation of the answer helps to analyze the qualitative behavior of the profile of ρ_+. Clearly, the truncated gaussian distribution which enters the expectation values $\langle\langle\cdots\rangle\rangle$ in (4.24) favors $p \approx 0$ if $\alpha < 0 < \beta$, $p \approx \alpha$ for $\alpha > 0$ and $p \approx \beta$ for $\beta < 0$. Thus,

$$\begin{aligned} &\rho_+ \sim 0, && x_3 > 0, \\ &\rho_+ \sim -2x_3, && 0 > x_3 > -\frac{1}{2}a, \\ &\rho_+ \sim a, && -\frac{1}{2}a > x_3. \end{aligned} \tag{4.26}$$

This behavior agrees with the expectations about the tilted D1-string suspended between two D3-branes separated by a distance $|a|$. The eigenvalue ρ_+ corresponds roughly to the the transverse coordinate of the D1 string, that runs from a at large negative x_3 to 0 at large positive x_3. In between the linear behavior of the Higgs field corresponds to the D1 string tilted at the critical angle. Indeed, for large $a \gg 1$, in the region $0 > x_3 > -\frac{1}{2}a$ this solution looks very similar to that of a single fluxon [11].

For future reference let us present the expression for another eigenvalue of $\langle 0|\Phi|0\rangle$, ρ_-:

$$\rho_- = \frac{2x_3(2x_3+a)M + M^2 - (2x_3+a)^2 - e^{-2x_3 a - a^2/2}}{M(2x_3 + a - 2x_3 M)},$$

$$\text{where} \qquad M = e^{-2x_3 a - a^2/2} + (2x_3 + a)\int_0^a e^{-2x_3 p - p^2/2}\, dp, \tag{4.27}$$

At this point, however, we should warn the reader that only the eigenvalues of the full, $2\infty \times 2\infty$ operator Φ should be identified with the D-brane profile. The components $\rho_\pm$ do not actually coincide with any of them. The eigenvalues of Φ, as it follows from the representation (4.8), are located between 0 and a, which is also what we expect from the dual D-brane picture [23].

5. Conclusions

In this paper we have presented a rather complete description of the classical, soliton, solutions of co-dimension two in non-commutative gauge theory. We showed that the non-commutative gauge theory contains the classical and quantum dynamics of all U(N) gauge theories and that classical solutions are labeled by the rank of the gauge group and the magnetic charge. We presented many examples of BPS and non-BPS solutions that can be constructed from the basic set of solutions when other matter fields are turned on. The BPS solutions describe various D-1 strings attached or piercing D3-branes. We analzed how the non-BPS solutions are unstable.

In addition we gave an explicit construction of a (localized in 3 dimensions) U(2) monopole, which has an intricate and interesting structure that corresponds precisely to the picture of a monopole as being a finite D1 string attached to two separated D3-branes.

The various solitons we have analyzed should have an interesing S-dual description in terms of fundamental strings, presumable in non-commutative open string theory. For example, if we wrap the non-BPS fluxon (with constant Φ) around a circle in the commutative (x_3) direction it should correspond in the strong coupling limit to a fundamental closed string wound around the circle. The instability of the fluxon to spread out over all the non-commutative space, should be the S-dual of the transition of the closed string to an open string on the brane, which can then dissipate.

Finally, we set up the machinery to derive an exact analytic soltution of 2-dimensional non-commutative gauge theory. It would be of great interest to complete this construction.

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